Mehler-Fock transform for tempered Boehmians

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Abstract

Boehmians are classified as classes of distribution spaces. Beginning with the definition of tempered Boehmians, in this paper we investigate the Mehler-Fock transformation of tempered Boehmians.

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1 Introduction

One of the youngest generalization of generalized function is Boehmians, which is motivated by regular operators, introduced by Boehme [1] . One may see several papers which relate the extension of the Boehmians in different classes, viz. tempered Boehmians and ultra Boehmians to the integral transformations and in other applications. In this paper the authors have attempted to extend the investigations of their paper [2] to the tempered Boehmians.

The Mehler- Fock transform and its inversion is defined as [7, p.149]

$$M[f(x)] = F(r) = \int_1^\infty P_{-\frac{1}{2}+ir}(x) f(x) dx \qquad , \quad r > 0 \tag{1}$$

i.e.

$$f(x) = \int_0^\infty r \tanh(\pi r) P_{-\frac{1}{2}+ir}(x) F(r) dr \qquad , \quad x > 1 \tag{2}$$

The generalization of the Mehler-Fock transform [5, p.343] is

$$F(r) = \int_0^\infty f(x) P^{m,n}_{-\frac{1}{2}+ir}(\cosh x) \sinh x dx \qquad , \tag{3}$$

where $P^{m,n}_{-\frac{1}{2}+ir}(\cosh x)$ is the generalized Legendre function, which is defined for complex values of the parameters k, m, n by

$$P_k^{m,n}(z) = \frac{(z+1)^{\frac{n}{2}}}{\Gamma(1-m)(z-1)^{\frac{m}{2}}} \, {}_2F_1\left[k + \frac{n-m}{2} + 1; -k + \frac{n-m}{2}; 1-m; \frac{1-z}{2}\right] \ , \tag{4}$$

for complex z not lying on the cross-cut along the real x - axis from 1 to $-\infty$.

The inversion formula for (3) is

$$f(x) = \int_0^\infty \chi(r) P^{m,n}_{-\frac{1}{2}+ir}(\cosh x) F(r) dr \qquad , \tag{5}$$

where

$$\chi(r) = \Gamma\left(\frac{1-m+n}{2} + ir\right)\Gamma\left(\frac{1-m+n}{2} - ir\right)$$
$$\times\Gamma\left(\frac{1-m-n}{2} + ir\right)\Gamma\left(\frac{1-m-n}{2} - ir\right)\left[\Gamma(2ir)\Gamma(-2ir)\pi 2^{n-m-2}\right]^{-1} \ . \tag{6}$$

When $m = n$, formulae given by (3) and (5) are reduced to the following

$$F(r) = \int_0^\infty f(\alpha) P^{m,n}_{-\frac{1}{2}+ir}(\cosh\alpha) \sinh\alpha \ d\alpha \quad , \tag{7}$$

and

$$f(\alpha) = \int_0^\infty r \tanh(\pi r) P_{-\frac{1}{2}+ir}(\cosh\alpha) F(r) dr \qquad . \tag{8}$$

Whereas, for $m = n = 0$, (3) and (5) reduce to (1) and (2), respectively.

The Parseval relation for the Mehler-Fock transform [6, pp.393-394] is written as

$$\int_0^\infty r\tanh(\pi r)F(r)G(r)dr = \int_1^\infty f(x)g(x)dx \quad . \tag{9}$$

The convolution for the Mehler-Fock transform is [7, p.237]

$$P_{-i\tau-\frac{1}{2}}\{(\mu f \overset{\wedge}{*} \mu g)\} = \frac{\pi^{\frac{3}{2}}}{\sqrt{2}\cosh(\pi\tau)}\, P_{i\tau-\frac{1}{2}}\{f\}P_{i\tau-\frac{1}{2}}\{g\} \quad . \tag{10}$$

where $\tau \in \mathbb{R}, \mu(x) = H(x-1)$.

Another convenient form of (10) is

$$M[f * g] = M[f]\cdot M[g] \quad . \tag{11}$$

Asymptotically, with respect to variables t and r, Equation (4) is expressed [5, p.345] as

$$P^{m,n}_{-\frac{1}{2}+ir}(\cosh t) = \begin{cases} O(t^{-\mathrm{Re}m}) & ,t \to 0+ \\ O(e^{-(1/2)t}) & ,t\to\infty \end{cases} , \tag{12}$$

$$P^{m,n}_{-\frac{1}{2}+ir}(\cosh t) = \begin{cases} O(1) & ,r\to 0+ \\ 2^{\frac{1}{2}(n-m-1)}\pi^{-1/2}(\sinh t)^{-1/2} & \\ \times(ir)^{m-1/2}\{e^{irt}+ie^{-i(m\pi+rt)}+O(r^{-1})\} & ,r\to\infty \end{cases} , \tag{13}$$

The distributional generalized Mehler-Fock transform F of the function $f(\cdot)$ is [5, p.346]

$$F(r) := \left\langle f(t), P^{m,n}_{-\frac{1}{2}+ir}(\cosh t)\right\rangle \quad , r \geq 0 \quad , \tag{14}$$

where $f \in M'^{\alpha}_{\beta}(I), \alpha \geq \mathrm{Re}(m)$ and $\beta \leq 1/2$.The space $M'^{\alpha}_{\beta}(I)$ is the dual of the testing function space$M^{\alpha}_{\beta}(I)$ [5, p.345] , which is the collection of all infinitely differentiable complex-valued function φ defined on an open interval I such that for every non negative integer k, real numbers $\alpha \geq \mathrm{Re}(m)$ and $\beta \leq 1/2$,

$$\gamma_k(\varphi) = \sup_{0<t<\infty} \left| \zeta(t) \bigtriangledown_t^k \varphi(t) \right| < \infty \quad , \tag{15}$$

where

$$\bigtriangledown_t = \left(D_t^2 + (\coth t) D_t + \frac{m^2}{2(1-\cosh t)} + \frac{n^2}{2(1+\cosh t)} \right) \quad , \tag{16}$$

and

$$\varsigma(t) = \varsigma_{a,b}(t) \ = \begin{cases} O(t^\alpha) & , t \to 0+ \\ \\ O(e^{\beta t}) & , t \to \infty \end{cases} \quad . \tag{17}$$

The topology over the space $M_\beta^\alpha(I)$ is generated by separating the collection of seminorms $\{\gamma_k\}_{k=0}^\infty$. For $r \geq 0$, the generalized Legendre function $P_{-\frac{1}{2}+ir}^{m,n}(\cosh t) \in M_\beta^\alpha(I)$ and satisfies

$$D^2 y + (\coth t) D y + \left[\frac{m^2}{2(1-\cosh t)} + \frac{n^2}{2(1+\cosh t)} + \left(r^2 + \frac{1}{4} \right) \right] y = 0 \quad . \tag{18}$$

Therefore,

$$\bigtriangledown_t P_{-\frac{1}{2}+ir}^{m,n}(\cosh t) = -\left(r^2 + \frac{1}{4} \right) P_{-\frac{1}{2}+ir}^{m,n}(\cosh t) \qquad . \tag{19}$$

The differentiability of the Mehler-Fock transform $F(r)$, where $f \in M_\beta^\alpha(I), \alpha \geq \mathrm{Re}(m)$ and $\beta \leq 1/2$, is [5, p.347]

$$F'(r) := \left\langle f(t), \left(\frac{\partial}{\partial r} \right) P_{-\frac{1}{2}+ir}^{m,n}(\cosh t) \right\rangle \qquad . \tag{20}$$

The operator transformation formula for $f \in M_\beta^\alpha(I)$ and $\varphi \in M_\beta^\alpha(I)$, respectively, are defined by

$$\left\langle (\bigtriangledown_t^*)^k f(t), \varphi(t) \right\rangle = \quad \left\langle f(t), (\bigtriangledown_t^*)^k \varphi(t) \right\rangle \qquad , \tag{21}$$

and

$$M\left[(\bigtriangledown_t^*)^k f(t)\right] = (-1)^k \left(r^2 + \frac{1}{4}\right)^k M[f(t)] \quad . \tag{22}$$

The Parseval relation (9) is defined for the generalized Mehler-Fock transformation in the following forms

$$\langle F(r), G(r)\rangle = \langle f(x), g(x)\rangle \quad , \tag{23}$$

and

$$\langle F(r), g(x)\rangle = \langle G(r), f(x)\rangle \quad , \tag{24}$$

2. Mehler-Fock Transformation of Tempered Boehmians

The pair of sequence (f_a, φ_a) is called a quotient of sequence, denoted by f_a/φ_a, whose numerator belongs to some set $\mathcal{A}$ and the denominator is a delta sequence such that

$$f_a * \varphi_b = f_b * \varphi_a \quad , \ \forall a, b \in \mathbb{N} \quad . \tag{25}$$

Two quotients of sequence f_a/φ_a and g_a/ψ_a are said to be equivalent if

$$f_a * \psi_a = g_a * \varphi_a \quad , \ \forall a \in \mathbb{N} \quad . \tag{26}$$

The equivalence classes are called the Boehmians. The space of Boehmians is denoted by β , an element of which is written as $x = f_a/\varphi_a$. Application of construction of Boehmians to function spaces with the convolution product yields various spaces of generalized functions. The spaces, so obtained, contain the standard spaces of generalized functions defined as dual spaces. For example, if $\mathcal{A} = C(\mathbb{R}^N)$ and a delta sequence defined as sequence of functions $\varphi_n \in \mathcal{D}$ such that

(i) $\int \varphi_a dx = 1 \quad , \ \forall a \in \mathbb{N}$

(ii) $\int |\varphi_a| \, dx \leq C \quad$, for some constant C and $\forall a \in \mathbb{N}$,

(iii) supp $\varphi_a(x) \to 0$, as $\ n \to \infty$

then the space of Boehmian that is obtained, contains properly the space of Schwartz distributions. Similarly, this space of Boehmians also contains properly the space of tempered distributions S' , when $\mathcal{A}$ is the space of slowly increasing functions with delta sequence. The Mehler-Fock transform of tempered Boehmian form a proper subspace of Schwartz distribution $\mathcal{D}'$. Boehmian space have two types of convergence, namely, the δ- and Δ- convergences, which are stated as:

(i) A sequence of Boehmians (x_a) in the Boehmian space B is said to be δ-convergent to a Boehmian x in B, which is denoted by $x_a \xrightarrow{\delta} x$ if there exists a delta sequence (δ_a) such that $(x_a * \delta_a), (x * \delta_a) \in \mathcal{A}, \forall a \in \mathbb{N}$ and $(x_a * \delta_k) \to (x * \delta_k)$ as $a \to \infty$ in $\mathcal{A}$, $\forall k \in \mathbb{N}$.

(ii) A sequence of Boehmians (x_a) in B is said to be Δ- convergent to a Boehmian x in B, denoted by $x_a \xrightarrow{\delta} x\ $ if there exists a delta sequence $(\delta_a) \in \Delta$ such that $(x_a - x) * \delta_a \in \mathcal{A}, \forall n \in \mathbb{N}$ and $(x_a - x) * \delta_a \to 0$ as $a \to \infty$ in $\mathcal{A}$.

For details of the properties and convergence of Boehmians one can refer to [3, 4]. We have employed following notations and definitions.

A complex valued infinitely differentiable function f , defined on $\mathbb{R}^N$, is called rapidly decreasing, if

$$\sup_{|\alpha| \leq m} \sup_{x \in R^N} \left(1 + x_1^2 + x_2^2 + \cdots + x_N^2\right)^m |D^\alpha f(x)| < \infty \qquad ,$$

for every non-negative integer m. Here $|\alpha| = |\alpha_1| + \cdots + |\alpha_N|$, and

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \qquad .$$

The space of all rapidly decreasing functions on $\mathbb{R}^N$ is denoted by S. The delta sequence, i.e., sequence of real valued functions $\varphi_1, \varphi_2, \ldots \in S$, is such that

(i) $\int \varphi_a dx = 1 \quad , \ \forall a \in \mathbb{N}$

(ii) $\int |\varphi_a| \, dx \leq C \quad$, for some constant C and $\forall a \in \mathbb{N}$,

(iii) $\lim\limits_{a\to\infty} \int_{\|x\|\geq\epsilon} \|x\|^k |\varphi_a| \, dx = 0$, for every $k \in \mathbb{N}, .\epsilon > 0$

If $\varphi \in S$ and $\int \varphi = 1$, then the sequence of functions φ_a is a delta sequence.

A complex-valued function f on $\mathbb{R}^N$ is called slowly increasing if there exists a polynomial p on $\mathbb{R}^N$ such that $f(x)/p(x)$ is bounded. The space of all increasing continuous functions on $\mathbb{R}^N$ is denoted by $\mathcal{I}$. If $f_a \in \mathcal{I}$, $\{\varphi_a\}$ is a delta sequence under usual notion, then the space of equivalence classes of quotients of sequence will be denoted by $\mathcal{B}_{\mathcal{I}}$, elements of which will be called tempered Boehmians.

For $F = [f_a/\varphi_a] \in \mathcal{B}_{\mathcal{I}}$, define $D^\alpha F = [(f_a * D^\alpha \varphi_a)/(\varphi_a * \varphi_a)]$. If F is a Boehmian corresponding to differentiable function, then $D^\alpha F \in \mathcal{B}_{\mathcal{I}}$.

If $F = [f_a/\varphi_a] \in \mathcal{B}_{\mathcal{I}}$ and $f_a \in S$, for all $a \in \mathbb{N}$, then F is called a rapidly decreasing Boehmian. The space of all rapidly decreasing Boehmian is denoted by $\mathcal{B}_S$. If $F = [f_a/\varphi_a] \in \mathcal{B}_{\mathcal{I}}$ and $G = [g_a/\gamma_a] \in \mathcal{B}_S$,then the convolution is

$$F * G = [(f_a * g_a)/(\varphi_a * \gamma_a)] \in \mathcal{B}_{\mathcal{I}} \quad .$$

The convolution quotient is denoted by f/φ and $\frac{f}{\varphi}$ denotes a usual quotient. Let $f \in \mathcal{I}$. Then the Mehler-Fock transformation of f, denoted as $M(f)$, is defined for distribution spaces (as in (23)) of slowly increasing function f in the following forms

$$\langle Mf, M\varphi \rangle = \langle f, \varphi \rangle \qquad , \ \varphi \in S$$

and

$$M[\varphi(x)] = \int\limits_0^\infty \varphi(x) P^{m,n}_{-\frac{1}{2}+ir}(\cosh x) \sinh x dx \quad .$$

Theorem 1. *If* $[f_a/\varphi_a] \in \mathcal{B}_{\mathcal{I}}$*, then the sequence* $M\{f_a\}$ *converges in* $\mathcal{D}'$*. Moreover, if* $[f_a/\varphi_a] = [g_a/\gamma_a]$*,then* $M\{f_a\}$ *and* $M\{g_a\}$ *converges to the same limit for the Mehler-Fock transformation of tempered Boehmians.*

Proof. Let $\varphi \in \mathcal{D}$ (testing function space) and $p \in \mathbb{N}$ be such that $M\varphi_p > 0$ on the support of φ. Since

$$f_a * \varphi_b = \quad f_b * \varphi_a \qquad , \ \forall a, b \in \mathbb{N}$$

thus,

$$M(\ f_a) \cdot M(\varphi_b) = \ M(\ f_b) \cdot M(\varphi_a) \qquad , \ \forall a, b \in \mathbb{N} \qquad .$$

We, thus, write the following

$$\langle\ M(\ f_a), M(\varphi_a)\rangle = \left\langle\ Mf_a, M\varphi_a \cdot \frac{M\varphi_p}{M\varphi_p}\right\rangle$$

$$= \left\langle\ Mf_a \cdot M\varphi_p, \frac{M\varphi_a}{M\varphi_p}\right\rangle$$

$$= \left\langle\ Mf_p \cdot M\varphi_a, \frac{M\varphi_a}{M\varphi_p}\right\rangle$$

$$= \left\langle\ Mf_p, \frac{M\varphi_a \cdot M\varphi_a}{M\varphi_p}\right\rangle \qquad .$$

Since the sequence $\left\{\frac{M\varphi_a \cdot M\varphi_a}{M\varphi_p}\right\}$ converges to $\frac{M\varphi_a}{M\varphi_p}$ in $\mathcal{D}$.Thus, this proves that the sequence $\{Mf_a, M\varphi_a\}$ converges in $\mathcal{D}$.This proves that Mf_a converges in $\mathcal{D}'$ (dual of space $\mathcal{D}$). Now to prove the second part of the theorem, we assume that $[f_a/\varphi_a] = [g_a/\gamma_a] \in \mathcal{B}_{\mathcal{I}}$. Define

$$h_a \ = \begin{cases} f_{\frac{a+1}{2}} * \gamma_{\frac{a+1}{2}} & \text{, if } a \text{ is odd} \\ g_{\frac{a}{2}} * \varphi_{\frac{a}{2}} & \text{, if } a \text{ is even} \end{cases} \qquad .$$

and

$$\delta_a = \begin{cases} \varphi_{\frac{a+1}{2}} * \gamma_{\frac{a+1}{2}}, & \text{if } a \text{ is odd,} \\ \varphi_{\frac{a}{2}} * \gamma_{\frac{a}{2}}, & \text{if } a \text{ is even.} \end{cases}$$

Then $[h_a/\delta_a] = [f_a/\varphi_a] = [g_a/\gamma_a]$. This proves that $M\{h_a\}$ converges in $\mathcal{D}'$. Moreover,

$$\begin{aligned} \lim_{a\to\infty} M\{h_{2a-1}\}M\{\varphi_a\} &= \lim_{a\to\infty} M\{f_a * \gamma_a\} \cdot M\{\varphi_a\} = \lim_{a\to\infty} M\{f_a\} \cdot M\{\gamma_a\} \cdot M\{\varphi_a\} \\ &= \lim_{a\to\infty} M\{f_a\} \cdot M\{\varphi_a\}. \end{aligned}$$

Thus, $M\{f_a\}$ and $M\{h_a\}$ have the same limit. Similarly, it can be shown that $M\{h_a\}$ and $M\{g_a\}$ will also have the same limit. This completes the proof of the theorem. □

Theorem 2. *Let $F = [f_a/\varphi_a] \in \mathcal{B}_{\mathcal{I}}$ and $G = [g_a/\gamma_a] \in \mathcal{B}_S$.*

Then (i) $M\left[(\bigtriangledown_t^*)^k f(x)\right] = (-1)^k \left(r^2 + \frac{1}{4}\right)^k M[f(x)]$ or $\frac{\partial^k F(r)}{\partial r} = (-1)^k \left(r^2 + \frac{1}{4}\right)^k F(r)$

(ii) $M(G)$ is an infinitely differentiable function

(iii) $M[F * G] = M[F]M[G]$

and (iv) $M(F) \cdot M(\varphi_p) = M(f_p)$, $p \in \mathbb{N}$.

Proof. (i)

$$\begin{aligned} \left(\frac{\partial^k F}{\partial r}\right) &= M\left[\frac{\left(\frac{\partial f_a}{\partial r} * \varphi_a\right)}{(\varphi_a * \varphi_a)}\right] = \lim_{a\to\infty} M\left(\frac{\partial f_a}{\partial r} * \varphi_a\right) \\ &= \lim_{a\to\infty} (-1)^k \left(r^2 + \frac{1}{4}\right)^k M f_a \cdot M\varphi_a = (-1)^k \left(r^2 + \frac{1}{4}\right)^k M(f_a) \end{aligned}$$

The last equality follows from Th.1 and also due to $[f_a/\varphi_a] = [(f_a * \varphi_a)/(\varphi_a * \varphi_a)]$.

(ii) Let $G = [g_a/\gamma_a] \in \mathcal{B}_S$ and let U be the bounded open subset of $\mathbb{R}^N$. Then there exists $a \in N$ such that $M\{\gamma_a\} > 0$ on U . We have, thus

$$M[G] = \lim_{a\to\infty} M\{g_a\} = \lim_{a\to\infty} \frac{M\{g_a\}M\{\gamma_p\}}{M\{\gamma_p\}}$$

$$= \lim_{a\to\infty} \frac{M\{g_p\}M\{\gamma_a\}}{M\{\gamma_p\}} = \frac{M\{g_p\}}{M\{\gamma_p\}} \lim_{a\to\infty} M\{\gamma_a\} = \frac{M\{g_p\}}{M\{\gamma_p\}} \quad \text{on } U \quad .$$

Since $\{g_p\}, M\{\gamma_p\} \in S$ and $M\{\gamma_p\} > 0$ on U, thus $M\{G\}$ is an infinitely differentiable function on U.

(iii) Let $F = [f_a/\varphi_a] \in \mathcal{B}_{\mathcal{I}}$ and $G = [g_a/\gamma_a] \in \mathcal{B}_S$. If $\varphi \in \mathcal{D}$, then there exists $p \in N$ such that $M\{\gamma_p\} > 0$ on the support of φ. We have

$$M\{F * G\}\{\varphi\} = \lim_{a\to\infty} M(f_a * g_a)(\varphi)$$

$$= \lim_{a\to\infty} M(f_a g_a)(\varphi) = \lim_{a\to\infty} M(f_a)(Mg_a, \varphi)$$

$$= \lim_{a\to\infty} M(f_a)\left\{\frac{Mg_a \cdot M\gamma_p}{M\gamma_p} \cdot \varphi\right\} \qquad , \quad p \in \mathbb{N}$$

$$= \lim_{a\to\infty} M(f_a)\left\{\frac{Mg_p \cdot M\gamma_a}{M\gamma_p} \cdot \varphi\right\}$$

$$= \lim_{a\to\infty} M(f_a)\left\{\frac{Mg_p}{M\gamma_p} \cdot \varphi M(\gamma_a)\right\}$$

$$= \lim_{a\to\infty} M(f_a)\{M(G)\varphi M(\gamma_a)\} \qquad ; \quad \text{from (ii)}$$

$$= M(G) \lim_{a\to\infty} \{M(f_a)M(\gamma_a)\}(\varphi)$$

$$= M(G) \lim_{a\to\infty} M(f_a * \gamma_a)(\varphi)$$

$$= M(F)M(G)(\varphi) = M\{F\} \cdot M\{G\}\{\varphi\}$$

The last equality follows from Theorem 1 and due to the fact that $[f_a/\varphi_a] = [(f_a * \varphi_a)/(\varphi_a * \gamma_a)]$.

(iv) Let $\varphi \in \mathcal{D}$. Then

$$\{MF \cdot M\varphi_p\}\{\varphi\} = \{MF\}\{M(\varphi_p)\varphi\} \quad , \quad p \in \mathbb{N}$$

$$= \lim_{a\to\infty}\{Mf_a\}\{M(\varphi_p)\varphi\} = \lim_{a\to\infty}(Mf_a \cdot M\varphi_p)(\varphi)$$

$$= \lim_{a\to\infty}(Mf_p \cdot M\varphi_a)(\varphi) = \lim_{a\to\infty} Mf_p(M\varphi_a \cdot \varphi)$$

$$= (Mf_p)(\varphi) = Mf_p$$

We assume that, every distribution is the Mehler-Fock transform of a tempered Boehmian. Consequently, the Mehler-Fock transform of an arbitrary distribution can be defined as a tempered Boehmian. The theorem is, therefore, completely proved. □

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